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Sinc-Gauss Sampling Formula

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1. Introduction

Shannon's sampling theorem [5] is fundamental in the field of information processing. Let

$$B_\sigma = \{f \in L^2(\mathbf{R}) \mid |\omega| > \sigma \Rightarrow \hat{f}(\omega) = 0\}, \tag{1.1}$$

which denotes a set of band-limited functions, where $\sigma > 0$ and $\hat{f}$ is the Fourier transform of f. The sampling theorem states that an identity

$$f(x) = \sum_{k=-\infty}^{\infty} f(kh)\,\mathrm{sinc}(x/h - k) \quad (x \in \mathbf{R}) \tag{1.2}$$

is valid for $f \in B_\sigma$, where $h = \pi/\sigma$ and

$$\mathrm{sinc}(x) = \begin{cases} \dfrac{\sin(\pi x)}{\pi x} & (x \neq 0), \\ 1 & (x = 0). \end{cases} \tag{1.3}$$

The sampling formula (1.2) shows that the function f can be reconstructed from the sampled values $f(kh)$ $(k \in \mathbf{Z})$.

Recently, this sampling formula has been put to use for numerical computation. A naive formula

$$f(x) \approx \sum_{k=\lfloor x/h \rfloor - N}^{\lceil x/h \rceil + N} f(kh)\,\mathrm{sinc}(x/h - k), \tag{1.4}$$

however, requires a prohibitively large number of sampling points due to the fact that the sinc function does not decrease rapidly at infinity. To overcome this difficulty, the following two methods are contrived.

The first is to transform $f(x)$ so that the transformed function $f(\varphi(t))$ may decrease rapidly at infinity through an appropriate change of variable $x = \varphi(t)$. Then a simple formula

$$f(x) \approx \sum_{k=-N}^{N} f(\varphi(kh))\,\mathrm{sinc}(t/h - k) \quad (t = \varphi^{-1}(x)) \tag{1.5}$$

is applied to the transformed function. The truncation error incurred in this approximation is bounded by $\sum_{|k|>N} |f(\varphi(kh))|$. If $f(\varphi(t))$ decreases rapidly at infinity, say, exponentially, the truncation error decreases exponentially with respect to the number of sampled points $2N + 1$. Numerical methods based on such function approximation are often referred to as "Sinc numerical methods" [6, 7, 8].

The second is to use rapidly decreasing kernel functions. A typical formula under this category is

$$f(x) \approx (\mathcal{T}_{N,h}f)(x) := \sum_{k=\lfloor x/h \rfloor - N}^{\lceil x/h \rceil + N} f(kh)\,\mathrm{sinc}(x/h - k) \exp\left[-\frac{(x-kh)^2}{2r^2h^2}\right], \tag{1.6}$$

where r is a positive constant. Seeing that no standard name of this formula is found in the literature, we call this formula the Sinc-Gauss sampling formula. This formula is used by Wei et al. in numerical solution of partial differential equations [12, 13][1]. Qian et al. show that the error $\|f - \mathcal{T}_{N,h}f\|_\infty$ of the Sinc-Gauss sampling formula decreases exponentially with respect to N for $f \in B_\sigma$, and also demonstrate similar results about the approximation of the derivatives of f [1, 2, 3, 4]. In estimating the discretization error, they make use of the Fourier transform and the Parseval identity to exploit the band-limited condition. In Japan, as early as in 1975, H. Takahasi [9] proposed the Sinc-Gauss sampling formula above to apply Shannon's sampling formula to numerical analysis. He made an error analysis for holomorphic functions by using complex analysis. His analysis lacks, however, in mathematical rigor, although it captures the essential feature.

The objective of this paper is to provide a mathematically rigorous version of Takahasi's error analysis for the Sinc-Gauss sampling formula. Furthermore, we point out that the formula is applicable to a wider class of functions including unbounded ones on $\mathbf{R}$. Specifically, we estimate the error of the formula for those functions which are holomorphic on a band-shaped region on the complex plane

$$\mathcal{D}_d := \{z \in \mathbf{C} \mid |\mathrm{Im} z| \leq d\} \tag{1.7}$$

and satisfy

$$|f(z)| \leq A + B|z|^\alpha \quad (\forall z \in \mathcal{D}_d),$$

where $d > 0$, $A \geq 0$, $B \geq 0$, and $\alpha \geq 0$. Furthermore, we show that part of Qian et al.'s result for $f \in B_\sigma$ can be derived from ours as an immediate corollary. It is mentioned that a preliminary result for bounded functions (i.e., for the case of $B = 0$) is discussed in [10], and that the proofs of the present results can be found in [11].

The organization of this paper is as follows. In Section 2, we present our main results. In Section 3, we specialize our results to bounded functions, and explain the relationship to some results of Qian-Ogawa [4]. In Section 4 we show computational results.

[1]To be precise, Wei et al. set $\lfloor x/h \rfloor + N$ as the upper bound of the sum, whereas we use $\lceil x/h \rceil + N$ for symmetry. This does not affect the following argument.

2. Main Results

For nonnegative integer m and positive numbers $r, h > 0$, we define operators $\mathcal{G}_h^{(m)}, \mathcal{T}_{N,h}^{(m)}$ approximating the m-th order derivative $f^{(m)}$ of a function f as

$$\left(\mathcal{G}_h^{(m)} f\right)(x) := \sum_{k=-\infty}^{\infty} f(kh) \frac{\mathrm{d}^m}{\mathrm{d}x^m}\left[\operatorname{sinc}(x/h - k) \exp\left[-\frac{(x-kh)^2}{2r^2h^2}\right]\right], \tag{2.1}$$

$$\left(\mathcal{T}_{N,h}^{(m)} f\right)(x) := \sum_{k=\lfloor x/h \rfloor - N}^{\lceil x/h \rceil + N} f(kh) \frac{\mathrm{d}^m}{\mathrm{d}x^m}\left[\operatorname{sinc}(x/h - k) \exp\left[-\frac{(x-kh)^2}{2r^2h^2}\right]\right], \tag{2.2}$$

where sinc is the function defined in (1.3). Note that (2.2) with $m = 0$ coincides with (1.6). We call the formula given by $\mathcal{T}_{N,h}^{(m)} f$ the Sinc-Gauss sampling formula.

Let $\mathcal{D}_d$ be the band-shaped region defined in (1.7). In this section, we assume that $f : \mathcal{D}_d \to \mathbf{C}$ is a holomorphic function on $\mathcal{D}_d$ with $|f(z)| \leq A + B|z|^\alpha$ $(\forall z \in \mathcal{D}_d)$ for constants $A \geq 0$, $B \geq 0$ and $\alpha \geq 0$. The error of the formula will be measured by the supremum of the absolute value of $f(x) - \left(\mathcal{T}_{N,h}^{(m)} f\right)(x)$ over a finite interval $[-L, L]$ for $L > 0$.

First, the discretization error of the Sinc-Gauss sampling formula is estimated as follows.

Lemma 2.1 (Discretization error). Let $d > 0$. Let $f : \mathcal{D}_d \to \mathbf{C}$ be a holomorphic function on $\mathcal{D}_d$ with $|f(z)| \leq A + B|z|^\alpha$ $(\forall z \in \mathcal{D}_d)$ for constants $A \geq 0$, $B \geq 0$ and $\alpha \geq 0$. Let $m \in \mathbf{Z}_+$, $L > 0$, $r > 0$, and $h > 0$ with $h \leq 2\pi d/\log 2$. Then we have

$$\begin{aligned}\sup_{-L \leq x \leq L} \left| f^{(m)}(x) - \left(\mathcal{G}_h^{(m)} f\right)(x) \right| &\leq \exp\left(-\frac{\pi d}{h} + \frac{d^2}{2r^2h^2}\right) \\ &\cdot C_0 \left[C_1 C_3 \sqrt{2\pi} + C_2 C_3 2^{\frac{\alpha+1}{2}} \Gamma\left(\frac{\alpha+1}{2}\right) \right. \\ &\left. + C_1 2^{\frac{2m+1}{2}} \Gamma\left(\frac{m+1}{2}\right) + C_2 2^{\frac{\alpha+2m+1}{2}} \Gamma\left(\frac{\alpha+m+1}{2}\right) \right],\end{aligned}$$

where

$$C_0 = \frac{2\pi^{m-1}(m+3)!\, r}{h^{m-1}} \left(1 + \left(\frac{\sqrt{2}}{rh}\right)^m\right) \left(\frac{1}{d} + \frac{1}{d^{m+1}}\right), \tag{2.3}$$

$$C_1 = A + 2^\alpha B(L+d)^\alpha, \tag{2.4}$$

$$C_2 = 2^\alpha B(rh)^\alpha, \tag{2.5}$$

$$C_3 = 2 + \left(\frac{\sqrt{2}d}{rh}\right)^m. \tag{2.6}$$

Second, the truncation error of the Sinc-Gauss sampling formula is estimated as follows.

Lemma 2.2 (Truncation error). Let $d > 0$. Let $f : \mathcal{D}_d \to \mathbf{C}$ be a holomorphic function on $\mathcal{D}_d$ with $|f(z)| \leq A + B|z|^\alpha$ $(\forall z \in \mathcal{D}_d)$ for constants $A \geq 0$, $B \geq 0$ and $\alpha \geq 0$. Let

$m \in \mathbf{Z}_+$, $L > 0$, $r > 0$, and $h > 0$. If $N \geq \max\left\{2,\ mr/\sqrt{2},\ \sqrt{\lceil\alpha\rceil}r + 1\right\}$, we have

$$\sup_{-L\leq x\leq L}\left|\left(\mathcal{G}_h^{(m)} f\right)(x) - \left(\mathcal{T}_{N,h}^{(m)} f\right)(x)\right| \leq C_0'(C_1' + C_2')\exp\left[-\frac{(N-1)^2}{2r^2}\right],$$

where

$$C_0' = \frac{2\,m!\,e^{\pi}e^{\frac{3}{2r^2}}r^2}{N(N-1)h^m\pi}, \tag{2.7}$$

$$C_1' = A + 2^{\alpha}B[(L+h)^{\alpha} + 2^{\alpha}h^{\alpha}], \tag{2.8}$$

$$C_2' = 2^{2\alpha}Bh^{\alpha}(\lceil\alpha\rceil + 1)!!\max\{(N-1)^{\lceil\alpha\rceil}, r^{\lceil\alpha\rceil}\}. \tag{2.9}$$

From the lemmas above, we can derive the following error estimate by setting h and r appropriately for a given N.

Theorem 2.3 (Error of the Sinc-Gauss sampling formula). Let $d > 0$. Let $f : \mathcal{D}_d \to \mathbf{C}$ be a holomorphic function on $\mathcal{D}_d$ with $|f(z)| \leq A + B|z|^{\alpha}$ ($\forall z \in \mathcal{D}_d$) for constants $A \geq 0$, $B \geq 0$ and $\alpha \geq 0$. Let $m \in \mathbf{Z}_+$ and $L > 0$. For a positive integer N, define h and r as

$$h = \frac{d'}{N}, \quad r = \sqrt{\frac{N}{\pi}} \tag{2.10}$$

with an arbitrary constant d' satisfying $0 < d' \leq d$. Then we have

$$\begin{aligned}&\sup_{-L\leq x\leq L}\left|f^{(m)}(x) - \left(\mathcal{T}_{N,h}^{(m)} f\right)(x)\right| \\ &= \mathrm{O}\left(N^{2m-\min\{1/2,\ 1-\lceil\alpha\rceil+\alpha\}}\exp\left(-\frac{\pi N}{2}\right)\right) \quad (N \to \infty).\end{aligned}$$

Proof. If N is sufficiently large, the assumptions in Lemmas 2.1 and 2.2,

$$h \leq 2\pi d/\log 2, \quad N \geq \max\left\{2,\ mr/\sqrt{2},\ \sqrt{\lceil\alpha\rceil}r + 1\right\}$$

are satisfied under (2.10). We apply the lemmas to the right hand side of the inequality

$$\begin{aligned}&\sup_{-L\leq x\leq L}\left|f^{(m)}(x) - \left(\mathcal{T}_{N,h}^{(m)} f\right)(x)\right| \\ &\leq \sup_{-L\leq x\leq L}\left|f^{(m)}(x) - \left(\mathcal{G}_h^{(m)} f\right)(x)\right| + \sup_{-L\leq x\leq L}\left|\left(\mathcal{G}_h^{(m)} f\right)(x) - \left(\mathcal{T}_{N,h}^{(m)} f\right)(x)\right|.\end{aligned}$$

The estimate in Lemma 2.1 remains valid when d is replaced by d'. With h and r in (2.10) we have

$$\exp\left(-\frac{\pi d'}{h} + \frac{d'^2}{2r^2h^2}\right) = \exp\left(-\frac{\pi N}{2}\right),$$

$$\exp\left[-\frac{(N-1)^2}{2r^2}\right] = \mathrm{O}\left(\exp\left(-\frac{\pi N}{2}\right)\right) \quad (N \to \infty).$$

Furthermore, the orders of $C_0, \dots, C_3$ in Lemma 2.1 and C_0', C_1', C_2' in Lemma 2.2 as $N \to \infty$ are estimated as follows:

$$C_0 = \mathrm{O}\left(N^{\frac{3m-1}{2}}\right),\ C_1 = \mathrm{O}(1),\ C_2 = \mathrm{O}\left(N^{-\frac{\alpha}{2}}\right),\ C_3 = \mathrm{O}\left(N^{\frac{m}{2}}\right),$$
$$C_0' = \mathrm{O}\left(N^{m-1}\right),\ C_1' = \mathrm{O}(1),\ C_2' = \mathrm{O}\left(N^{\lceil\alpha\rceil-\alpha}\right).$$

Thus we obtain the claim of the theorem. □

The error estimate in Theorem 2.3 presupposes approximation of $f(x)$ at a single point x and, accordingly, expresses the error bound in terms of the number $2N+1$ of the sampling points required for a single point. In some situations, however, it is more natural to consider approximation over a finite interval $[-L, L]$ with $L > 0$. This is the case, for instance, in applications to differential equations. In such a case it is more appropriate to express the error bound in terms of the number

$$M = 2\left(\frac{L}{d'} + 1\right) N \tag{2.11}$$

of the sampling points needed for the approximation over the entire interval, rather than at a single point, where d' is in (2.10). In accordance with this, Theorem 2.3 can be recast into the following form.

Corollary 2.4. Let $d > 0$. Let $f : \mathcal{D}_d \to \mathbf{C}$ be a holomorphic function on $\mathcal{D}_d$ with $|f(z)| \le A + B|z|^\alpha$ ($\forall z \in \mathcal{D}_d$) for constants $A \ge 0$, $B \ge 0$ and $\alpha \ge 0$. Let $m \in \mathbf{Z}_+$ and $L > 0$. For a positive integer N, define h and r as (2.10), and M as (2.11). Then we have

$$\sup_{-L \le x \le L} \left| f^{(m)}(x) - \left(\mathcal{T}_{N,h}^{(m)} f\right)(x) \right|$$
$$= \mathrm{O}\left(M^{2m - \min\{1/2,\ 1-\lceil\alpha\rceil+\alpha\}} \exp\left(-\frac{\pi d'}{4(d'+L)} M \right) \right) \quad (M \to \infty).$$

3. Error Estimates for Bounded Functions

In this section, we present the error estimate for holomorphic functions $f : \mathcal{D}_d \to \mathbf{C}$ on $\mathcal{D}_d$ with $|f(z)| \le A$ ($\forall z \in \mathcal{D}_d$) for a constant $A \ge 0$, and discuss its relationship to the estimate of Qian-Ogawa [4]. For bounded functions it is possible to consider supremum error bounds over the entire real number $\mathbf{R}$. The error estimates over $\mathbf{R}$ can be obtained easily from our results in Section 2 by setting $B = 0$, $\alpha = 0$ and letting $L \to \infty$. We set $\|g\|_\infty := \sup_{-\infty < x < \infty} |g(x)|$ for a function g on $\mathbf{R}$.

3.1. **Error Estimates.** Letting $B = 0$, $\alpha = 0$, $L \to \infty$ in Section 2, we obtain the following lemmas and theorem.

Lemma 3.1 (Discretization error). Let $d > 0$. Let $f : \mathcal{D}_d \to \mathbf{C}$ be a holomorphic function on $\mathcal{D}_d$ with $|f(z)| \le A$ ($\forall z \in \mathcal{D}_d$) for a constant $A \ge 0$. Let $m \in \mathbf{Z}_+$, $r > 0$, and $h > 0$

with $h \leq 2\pi d/\log 2$. Then we have

$$\left\| f^{(m)} - \mathcal{G}_h^{(m)} f \right\|_\infty \leq \exp\left(-\frac{\pi d}{h} + \frac{d^2}{2r^2h^2}\right) \cdot A \left[\frac{2\pi^{m-1}(m+3)!\, r}{h^{m-1}} \left(1 + \left(\frac{\sqrt{2}}{rh}\right)^m\right)\left(\frac{1}{d} + \frac{1}{d^{m+1}}\right)\right] \cdot \left[\sqrt{2\pi}\left(2 + \left(\frac{\sqrt{2}d}{rh}\right)^m\right) + 2^{\frac{2m+1}{2}}\Gamma\left(\frac{m+1}{2}\right)\right].$$

Lemma 3.2 (Truncation error). Let $d > 0$. Let $f : \mathcal{D}_d \to \mathbf{C}$ be a holomorphic function on $\mathcal{D}_d$ with $|f(z)| \leq A$ ($\forall z \in \mathcal{D}_d$) for a constant $A \geq 0$. Let $m \in \mathbf{Z}_+$, $r > 0$, and $h > 0$. If $N \geq \max\{2,\ mr/\sqrt{2}\}$, we have

$$\left\| \mathcal{G}_h^{(m)} f - \mathcal{T}_{N,h}^{(m)} f \right\|_\infty \leq \frac{2A\, m!\, e^{\pi} e^{\frac{3}{2r^2}} r^2}{N(N-1)h^m \pi} \exp\left[-\frac{(N-1)^2}{2r^2}\right].$$

Theorem 3.3 (Error of the Sinc-Gauss sampling formula). Let $d > 0$. Let $f : \mathcal{D}_d \to \mathbf{C}$ be a holomorphic function on $\mathcal{D}_d$ with $|f(z)| \leq A$ ($\forall z \in \mathcal{D}_d$) for a constant $A \geq 0$. Let $m \in \mathbf{Z}_+$. For a positive integer N, define h and r as

$$h = \frac{d'}{N}, \quad r = \sqrt{\frac{N}{\pi}} \tag{3.1}$$

with an arbitrary constant d' satisfying $0 < d' \leq d$. Then we have

$$\left\| f^{(m)} - \mathcal{T}_{N,h}^{(m)} f \right\|_\infty = \mathrm{O}\left(N^{2m-1/2} \exp\left(-\frac{\pi N}{2}\right)\right) \quad (N \to \infty).$$

3.2. **Relationship to Qian-Ogawa's Result.** We investigate the relationship between the result of Qian-Ogawa [4] and our Theorem 3.3 in Section 3.1. The following theorem is an immediate corollary of Corollary 3.1 of [4], where B_σ is defined as (1.1).

Theorem 3.4 ([4]). Let $f \in B_\sigma$ and $0 < h < \pi/\sigma$. For $N > 2$, define $r = \sqrt{(N-2)/(\pi - h\sigma)}$. Then we have

$$\left\| f^{(m)} - \mathcal{T}_{N,h}^{(m)} f \right\|_\infty = \mathrm{O}\left(\frac{1}{\sqrt{N-2}} \exp\left[-\frac{(\pi - h\sigma)(N-2)}{2}\right]\right) \quad (N \to \infty). \tag{3.2}$$

The objective of this section is to demonstrate how (3.2) with $m = 0$ can be derived from our result of Section 3.1. In the case of $m \geq 1$ we also derive a weaker result[2]

$$\left\| f^{(m)} - \mathcal{T}_{N,h}^{(m)} f \right\|_\infty = \mathrm{O}\left((N-2)^{(m-1)/2} \exp\left[-\frac{(\pi - h\sigma)(N-2)}{2}\right]\right) \quad (N \to \infty). \tag{3.3}$$

[2]The estimate (3.2) does not seem to be derived in the case of $m \geq 1$ from our results. This is because our estimate of the discretization error is considered under a more general condition, and is necessarily weaker.

First we note the following fact, which may be regarded as a part of the Paley-Wiener theorem.

Lemma 3.5. If $f \in B_\sigma$, then f is holomorphic on $\mathbf{C}$ and there exists a constant $A' \geq 0$ such that

$$|f(z)| \leq A' \exp(\sigma|\mathrm{Im} z|) \quad (z \in \mathbf{C}). \tag{3.4}$$

Proof. Denote the Fourier transform of f by $\hat{f}$. By $f \in B_\sigma$, we have $\hat{f} \in L^2(\mathbf{R})$ and

$$f(z) = \frac{1}{\sqrt{2\pi}} \int_{-\sigma}^{\sigma} \hat{f}(\omega) \exp(\mathrm{i}z\omega) \mathrm{d}\omega. \tag{3.5}$$

Since the interval of integration is finite, we can exchange the differentiation and integration. Therefore f is holomorphic on $\mathbf{C}$.

Next, again by (3.5), we have

$$\begin{aligned} |f(\xi + \mathrm{i}\eta)| &\leq \frac{1}{\sqrt{2\pi}} \int_{-\sigma}^{\sigma} |\hat{f}(\omega)| |\exp(\mathrm{i}(\xi + \mathrm{i}\eta)\omega)| \mathrm{d}\omega \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\sigma}^{\sigma} |\hat{f}(\omega)| \exp(-\eta\omega) \mathrm{d}\omega \\ &\leq \exp(|\eta|\sigma) \cdot \frac{1}{\sqrt{2\pi}} \int_{-\sigma}^{\sigma} |\hat{f}(\omega)| \mathrm{d}\omega \end{aligned}$$

for $\xi, \eta \in \mathbf{R}$, and therefore (3.4) by setting $A' = (2\pi)^{-1/2} \int_{-\sigma}^{\sigma} |\hat{f}(\omega)| \mathrm{d}\omega$, which is finite since

$$\left(\int_{-\sigma}^{\sigma} |\hat{f}(\omega)| \mathrm{d}\omega \right)^2 \leq \int_{-\sigma}^{\sigma} \mathrm{d}\omega \int_{-\sigma}^{\sigma} |\hat{f}(\omega)|^2 \mathrm{d}\omega < \infty.$$

□

Lemma 3.5 above implies the following, which states that our function class contains band-limited functions.

Lemma 3.6. Let $f \in B_\sigma$. For any $d > 0$, $f : \mathcal{D}_d \to \mathbf{C}$ is a holomorphic function on $\mathcal{D}_d$ with $|f(z)| \leq A$ $(\forall z \in \mathcal{D}_d)$, where

$$A = A' \exp(\sigma d) \tag{3.6}$$

with A' in (3.4).

This lemma enables us to apply Lemmas 3.1 and 3.2 to $f \in B_\sigma$. We take r as in Theorem 3.4 and assume that N is sufficiently large.

To estimate the discretization error, we set

$$d = h(N - 2)$$

and take A as (3.6). Then, by Lemma 3.1, we have

$$\left\| f^{(m)} - \mathcal{G}_h^{(m)} f \right\|_\infty \le A' \exp\left(\sigma d - \frac{\pi d}{h} + \frac{d^2}{2r^2h^2}\right) \cdot \underline{\left[\frac{2\pi^{m-1}(m+3)!\, r}{h^{m-1}}\left(1+\left(\frac{\sqrt{2}}{rh}\right)^m\right)\left(\frac{1}{d}+\frac{1}{d^{m+1}}\right)\right]} \cdot \underline{\left[\sqrt{2\pi}\left(2+\left(\frac{\sqrt{2}d}{rh}\right)^m\right)+2^{\frac{2m+1}{2}}\Gamma\left(\frac{m+1}{2}\right)\right]}.$$

The degree of the underlined part with respect to $N-2$ is $(m-1)/2$. The exponent of the remaining part is

$$\begin{aligned}\sigma d - \frac{\pi d}{h} + \frac{d^2}{2r^2h^2} &= -\frac{(\pi - h\sigma)d}{h} + \frac{d^2}{2r^2h^2} \\ &= -(\pi - h\sigma)(N-2) + \frac{(\pi - h\sigma)h^2(N-2)^2}{2(N-2)h^2} \\ &= -\frac{(\pi - h\sigma)(N-2)}{2}.\end{aligned}$$

Thus we obtain the following estimate:

$$\begin{aligned}&\left\| f^{(m)} - \mathcal{G}_h^{(m)} f \right\|_\infty \\ &= \mathrm{O}\left((N-2)^{(m-1)/2}\exp\left[-\frac{(\pi - h\sigma)(N-2)}{2}\right]\right) \quad (N \to \infty).\end{aligned} \tag{3.7}$$

To estimate the truncation error, we set $d = 1$ and $A = A'e^\sigma$ according to (3.6). Then, by Lemma 3.2, we have

$$\begin{aligned}\left\| \mathcal{G}_h^{(m)} f - \mathcal{T}_{N,h}^{(m)} f \right\|_\infty &\le \frac{2A'e^\sigma m!\, e^\pi e^{\frac{3}{2r^2}} r^2}{N(N-1)h^m\pi}\exp\left[-\frac{(N-1)^2}{2r^2}\right] \\ &\le \frac{2A'e^\sigma m!\, e^\pi e^{\frac{3}{2r^2}}}{h^m\pi}\,\frac{r^2}{(N-2)^2}\exp\left[-\frac{(N-2)^2}{2r^2}\right].\end{aligned}$$

The degree of $r^2/(N-2)^2$ with respect to $N-2$ is -1. Furthermore, $e^{\frac{3}{2r^2}} \to 1$ as $N \to \infty$. The exponent of the remaining part is

$$-\frac{(N-2)^2}{2r^2} = -\frac{(\pi - h\sigma)(N-2)}{2}.$$

Thus we obtain the following estimate:

$$\begin{aligned}&\left\| \mathcal{G}_h^{(m)} f - \mathcal{T}_{N,h}^{(m)} f \right\|_\infty \\ &= \mathrm{O}\left(\frac{1}{N-2}\exp\left[-\frac{(\pi - h\sigma)(N-2)}{2}\right]\right) \quad (N \to \infty).\end{aligned} \tag{3.8}$$

By (3.7) and (3.8), we have (3.3) in the case of $m \ge 0$ and (3.2) in the case of $m = 0$.

4. Numerical Experiments

In this section, we present computational results on Sinc-Gauss sampling formula for two types of functions: (i) rational functions

$$f_{\beta,d}(z) = \frac{z^{\beta+2}}{z^2 + d^2}$$

with $\beta \in \{-2, -1, 0, 1, 2\}$ and $d > 0$, and (ii) band-limited functions

$$f_l(z) = (\mathrm{sinc}(z))^l$$

with a positive integer l. The former is not band-limited, and the latter is included to confirm that the performance of the Sinc-Gauss sampling formula is essentially independent of the band-limited property of the functions to be approximated.

We consider errors on a finite interval $[-3, 3]$ (i.e., $L = 3$), which we evaluate numerically as the maximum of the errors at 6000 equally-spaced points in the interval. The relationship of the error against the number of sampling points will be presented in graphs. Specifically, the ordinates are the errors in logarithm,

$$\log_{10}\left(\sup_{-3\leq x\leq 3}\left|f(x) - \left(\mathcal{T}_{N,h}^{(m)} f\right)(x)\right|\right), \tag{4.1}$$

and the abscissae are N as well as $M = 2(3/d' + 1)N$ (with $L = 3$ in (2.11)), where M is indicated at the top.

According to our theoretical analysis summarized in Theorem 2.3, the error curves are expected to be almost linear, with the slope against N being

$$-\frac{\pi}{2}\log_{10}\mathrm{e} = -0.682\cdots. \tag{4.2}$$

This theoretical slope will be compared with the observed values, which we obtain from the computational results by the least square method.

The program for the computation is written in C. Our computer is SUN Blade 2000, whose environment is as follows: the operating system is Solaris 9, the CPU is UltraSPARC-III+(900MHz, 64bit) with 3 GB memory, the compiler is Sun Studio 11, in which "long double" is 128 bits wide.

4.1. Rational Functions.

For $\beta \in \{-2, -1, 0, 1, 2\}$ and $d > 0$, define $f_{\beta,d}$ as

$$f_{\beta,d}(z) = \frac{z^{\beta+2}}{z^2 + d^2} \quad (z \in \mathbf{C}). \tag{4.3}$$

Then $f_{\beta,d}$ is holomorphic on $\mathcal{D}_{d-\varepsilon}$ for ε with $0 < \varepsilon \ll d$, and satisfies

$$|f_{\beta,d}(z)| \leq \frac{\max\{d, d^{-1}\}}{\varepsilon}|z|^{\alpha} \quad (\forall z \in \mathcal{D}_{d-\varepsilon}),$$

where $\alpha = \max\{\beta, 0\}$. The Sinc-Gauss sampling formula is applied to $f_{\beta,d}$ for $\beta = -2, -1, 0, 1, 2$, $d = 10^{-i}$ $(i = 0, 1, 2)$, and $m = 0, 1, 2$. We set $\varepsilon = d/100$ and $h = (d-\varepsilon)/N$. Furthermore, in computing the slopes, we exclude the data for $N = 45$ and 50 to avoid the effect of rounding errors.

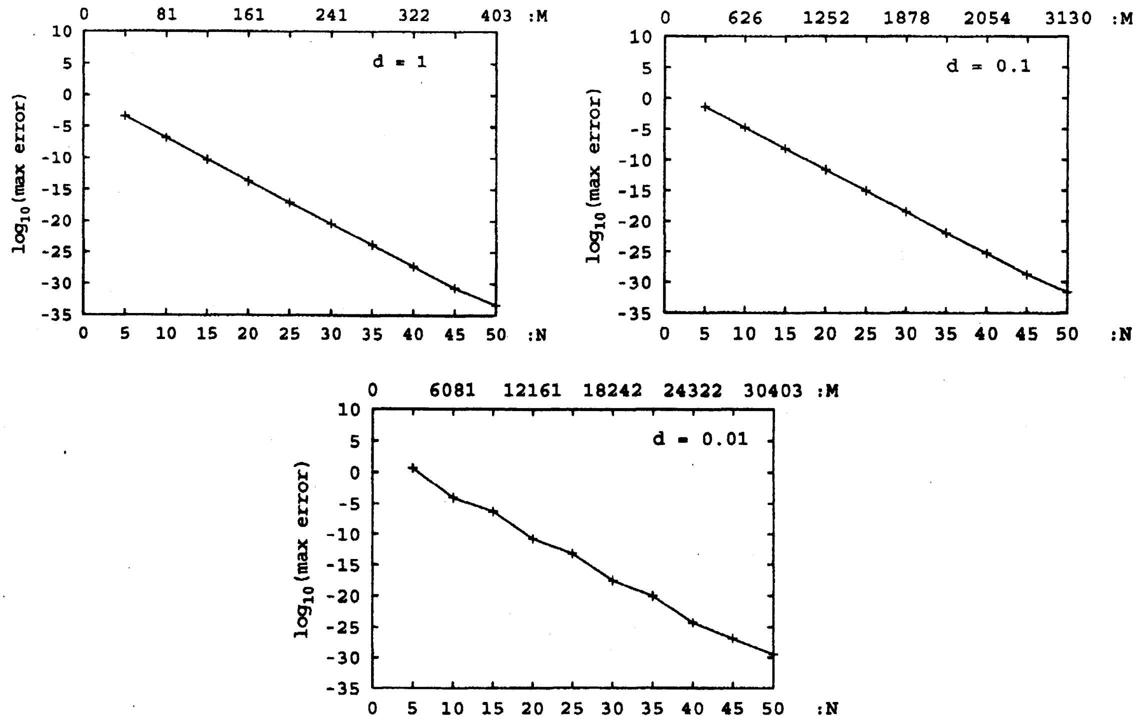

FIGURE 1. Errors for $f_{\beta,d}$ of (4.3) with $\beta = -2$ and for $m = 0$

TABLE 1. $\log_{10}$ (max error) for $f_{\beta,d}$ of (4.3) with $\beta = -2$

	$m=0$		$m=1$		$m=2$	
	$N=10$	$N=40$	$N=10$	$N=40$	$N=10$	$N=40$
$d=1.00$	−6.77	−27.28	−5.29	−25.19	−3.52	−22.80
$d=0.10$	−4.77	−25.32	−2.29	−22.20	0.48	−18.84
$d=0.01$	−4.09	−24.29	0.45	−19.55	4.43	−14.96

TABLE 2. $\log_{10}$ (max error) for $f_{\beta,d}$ of (4.3) with $\beta = -1$

	$m=0$		$m=1$		$m=2$	
	$N=10$	$N=40$	$N=10$	$N=40$	$N=10$	$N=40$
$d=1.00$	−6.94	−27.45	−5.31	−25.21	−3.58	−22.84
$d=0.10$	−5.96	−26.47	−3.31	−23.21	−0.59	−19.84
$d=0.01$	−6.09	−26.42	−1.31	−21.21	2.18	−17.18

TABLE 3. $\log_{10}$ (max error) for $f_{\beta,d}$ of (4.3) with $\beta = 0$

	$m=0$		$m=1$		$m=2$	
	$N=10$	$N=40$	$N=10$	$N=40$	$N=10$	$N=40$
$d=1.00$	−6.89	−27.35	−5.40	−25.26	−3.65	−22.87
$d=0.10$	−6.90	−27.39	−4.41	−24.26	−1.66	−20.87
$d=0.01$	−7.38	−28.14	−3.53	−23.52	0.34	−18.87

TABLE 4. $\log_{10}$ (max error) for $f_{\beta,d}$ of (4.3) with $\beta = 1$

	$m=0$		$m=1$		$m=2$	
	$N=10$	$N=40$	$N=10$	$N=40$	$N=10$	$N=40$
$d=1.00$	−6.94	−27.49	−5.41	−25.28	−3.61	−22.91
$d=0.10$	−6.91	−27.66	−4.41	−24.56	−1.60	−21.15
$d=0.01$	−6.92	−27.66	−3.40	−23.56	0.40	−19.15

TABLE 5. $\log_{10}$ (max error) for $f_{\beta,d}$ of (4.3) with $\beta = 2$

	$m=0$		$m=1$		$m=2$	
	$N=10$	$N=40$	$N=10$	$N=40$	$N=10$	$N=40$
$d=1.00$	−6.50	−27.23	−4.91	−25.07	−3.12	−22.67
$d=0.10$	−6.43	−27.19	−3.93	−24.08	−1.13	−20.68
$d=0.01$	−6.45	−27.19	−2.93	−23.09	0.87	−18.68

TABLE 6. $\log_{10}$ (max error)$/N$ for $f_{\beta,d}$ of (4.3) with $\beta = -2$

	$m=0$	$m=1$	$m=2$
$d=1.00$	−0.684	−0.660	−0.635
$d=0.10$	−0.686	−0.660	−0.636
$d=0.01$	−0.692	−0.664	−0.639

TABLE 7. $\log_{10}$ (max error)$/N$ for $f_{\beta,d}$ of (4.3) with $\beta = -1$

	$m=0$	$m=1$	$m=2$
$d=1.00$	−0.684	−0.660	−0.634
$d=0.10$	−0.685	−0.660	−0.634
$d=0.01$	−0.691	−0.660	−0.638

TABLE 8. $\log_{10}$ (max error)$/N$ for $f_{\beta,d}$ of (4.3) with $\beta = 0$

	$m=0$	$m=1$	$m=2$
$d=1.00$	−0.681	−0.658	−0.632
$d=0.10$	−0.683	−0.658	−0.632
$d=0.01$	−0.694	−0.663	−0.632

TABLE 9. $\log_{10}$ (max error)$/N$ for $f_{\beta,d}$ of (4.3) with $\beta = 1$

	$m=0$	$m=1$	$m=2$
$d=1.00$	−0.687	−0.660	−0.637
$d=0.10$	−0.694	−0.670	−0.646
$d=0.01$	−0.694	−0.670	−0.646

TABLE 10. $\log_{10}$ (max error)$/N$ for $f_{\beta,d}$ of (4.3) with $\beta = 2$

	$m=0$	$m=1$	$m=2$
$d=1.00$	−0.693	−0.670	−0.646
$d=0.10$	−0.694	−0.670	−0.646
$d=0.01$	−0.693	−0.669	−0.646

From Table 6–Table 10, we see that the experimental values of the slopes are close to the theoretical ones in (4.2). As m becomes larger, the slope tends to be larger than

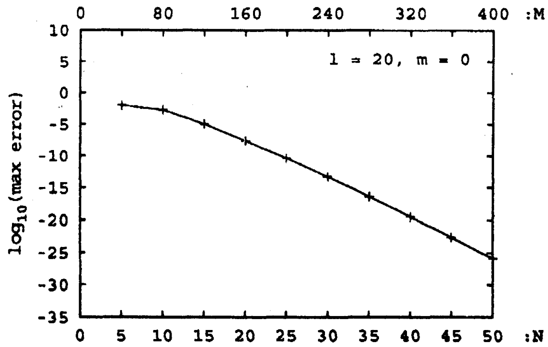

FIGURE 2. Errors for $f = \mathrm{sinc}^{20}$ with $m = 0$ and $d = 1$

the theoretical value. This may be because we only use $\exp(-(\pi/2)N)$ in Theorem 3.3 in computing the theoretical value, with the secondary factor $N^{2m-\min\{1/2,\ 1-\lceil\alpha\rceil+\alpha\}}$ disregarded.

Next, we consider the effect of m, the order of differentiation. By Theorem 3.3, we expect that $\log_{10}(\text{max error})$ will increase approximately by $2\log_{10} N$ if m increases by one. The results of Table 6–Table 10 agree with this expectation, whereas m is also included in the constant part independent of N in the estimate.

Next, we consider the effect of d, representing the location of the singular points. Noting the order with respect to N, we conclude that d does not affect the error. It is expected, however, that $\log_{10}(\text{max error})$ will increase approximately by $m+1$ if d is multiplied by $1/10$, due to the term $1/d^{m+1}$ in the estimate of Lemma 3.1. Computational results appear to support this observation.

Finally, we consider the effect of β. From the results, we see that β does not affect the errors substantially, which is theoretically appropriate.

4.2. **Band-limited Functions.** For a positive integer l, we define f_l as

$$f_l(z) = (\mathrm{sinc}(z))^l \quad (z \in \mathbf{C}). \tag{4.4}$$

Then we have $f_l \in B_{\pi l}$. The function f_l is holomorphic on $\mathbf{C}$ and satisfies

$$|f_l(z)| \le \max\left\{\left(\frac{e^{\pi d}}{\pi}\right)^l, e^{\pi l}\right\} \quad (z \in \mathcal{D}_d) \tag{4.5}$$

for arbitrary $d > 0$. Setting $h = 1/N$, we apply the Sinc-Gauss sampling formula to f_l for $l = 5, 10, 15, 20$ and $m = 0, 1, 2$. In computing the slopes, we exclude the data for $N = 5, 10, 45, 50$.

As to the effect of m on the errors, we see the same as in Section 4.1.

Next, we consider the slopes of the error curves in the graphs. The experimental values of the slopes are close to the theoretical ones in (4.2) when l is small (Table 12). In the case where l is large, however, this is not the case. This may be because the constant on the right hand side of (4.5) is large when l is large (note that $d = 1$), and the effect of the constant cannot be ignored.

TABLE 11. $\log_{10}(\text{max error})$ for $f = \text{sinc}^l$ with $d = 1$

	$m=0$		$m=1$		$m=2$	
	$N=10$	$N=40$	$N=10$	$N=40$	$N=10$	$N=40$
$l=5$	−5.18	−25.48	−3.67	−23.39	−1.98	−21.01
$l=10$	−3.99	−23.18	−2.39	−21.09	−0.81	−18.74
$l=15$	−3.18	−21.20	−1.66	−19.09	−0.11	−16.78
$l=20$	−2.70	−19.49	−1.23	−17.38	0.34	−15.09

TABLE 12. $\log_{10}(\text{max error})/N$ for $f = \text{sinc}^l$ with $d = 1$

	$m=0$	$m=1$	$m=2$
$l=5$	−0.680	−0.664	−0.645
$l=10$	−0.653	−0.638	−0.618
$l=15$	−0.618	−0.605	−0.583
$l=20$	−0.583	−0.568	−0.546

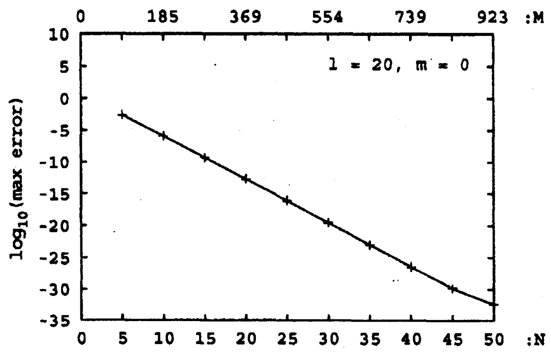

FIGURE 3. Errors for $f = \text{sinc}^{20}$ with $m = 0$ and $d = \pi^{-1}\log\pi$

TABLE 13. $\log_{10}(\text{max error})$ for $f = \text{sinc}^l$ with $d = \pi^{-1}\log\pi$

	$m=0$		$m=1$		$m=2$	
	$N=10$	$N=40$	$N=10$	$N=40$	$N=10$	$N=40$
$l=5$	−6.94	−27.69	−5.01	−25.15	−2.79	−22.32
$l=10$	−6.55	−27.25	−4.63	−24.72	−2.42	−21.49
$l=15$	−6.20	−26.83	−4.28	−24.29	−2.09	−21.16
$l=20$	−5.89	−26.41	−3.97	−23.88	−1.80	−20.62

Taking this fact into consideration, we apply the formula in the case of $d = \pi^{-1}\log\pi$, i.e., $h = (\pi N)^{-1}\log\pi$. The results of the experiments are presented in Fig. 3 and Table 14, which justify the above observation. The numerical results support our expectation that smaller width between neighboring sampling points yields better approximation for functions with strong vibration.

TABLE 14. $\log_{10}(\text{max error})/N$ for $f = \text{sinc}^l$ with $d = \pi^{-1}\log\pi$

	$m=0$	$m=1$	$m=2$
$l=5$	-0.690	-0.673	-0.656
$l=10$	-0.689	-0.672	-0.643
$l=15$	-0.688	-0.670	-0.644
$l=20$	-0.685	-0.668	-0.638

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